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LETTER TO THE EDITOR

Finding ground states in random-field Ising ferromagnets†

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Abstract. We discuss the similarities between random-field Ising ferromagnets and two-dimensional Ising spin glasses.

For a random-field Ising ferromagnet (RFIF) with sites $\{1, 2, \ldots, n\}$ we shall study the problem of obtaining a ground state, and the similarities with the same problem in two-dimensional (2D) Ising spin glasses in a zero field.

We are looking for the minimum of

$$H = -\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} J_{ij} S_i S_j - \sum_{i=1}^{n} F_i S_i$$

where the numbers J_{ij} are non-negative and the variables $S_i \in \{-1, 1\}$, for $1 \leqslant i \leqslant n$.

Barahona (1982) classified some of these models according to their computational complexity. The 2D problem is polynomially solvable (cf Bieche *et al* 1980). The RFIF problem is also polynomially solvable; in fact, Picard and Ratliff (1974) pointed out that this special case of a quadratic discrete optimisation problem can be reduced to a minimum-cut problem.

Barahona *et al* (1982) showed that the 2D problem can be reduced to a linear programming problem over a polyhedron defined by the frustrated contours. This has been done using the 'Chinese postman theorem' of Edmonds and Johnson (1973). This fact allows us to use linear programming duality to obtain information about the degeneracy of the ground state.

The purpose of this letter is to show that the same is true for the RFIF model. This is based in another classical theorem of polyhedral combinatorics, the 'max-flow min-cut theorem' of Ford and Fulkerson (1962).

In order to use the concept of 'frustration' introduced by Toulouse (1977) let us add one more spin $S_{n+1} \in \{-1, 1\}$, and we will look for the minimum of

$$H' = -\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} J_{ij} S_i S_j - \sum_{i=1}^{n} F_i S_i S_{n+1}.$$

It is clear that the minimum of H' is the same as the minimum of H.

Let us define the graph $G = (V, E)$ where $V = \{1, \ldots, n, n+1\}$, for $1 \leqslant i < j \leqslant n$ $(i, j) \in E$ if $J_{ij} > 0$, and $(i, n+1) \in E$ for $i = 1, \ldots, n$. Frustrated contours in G correspond to cycles in G containing exactly one edge of type $(i, n+1)$ with $F_i < 0$. It is

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well known that finding the minimum of H' is equivalent to obtaining a spin configuration that minimises the weight of the violated edges.

Let us introduce the variables x_{ij} for each $(i, j) \in E$, where

$$x_{ij} = \begin{cases} 1 & \text{if } (i, j) \text{ is violated} \\ 0 & \text{otherwise.} \end{cases}$$

Our problem is equivalent to minimising

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} J_{ij} x_{ij} + \sum_{i=1}^{n} |F_i| x_{i,n+1}$$

subject to

$$\begin{aligned}
&(a) \ \sum_{(i,j)\in C} x_{ij} \geq 1 \qquad \text{for each frustrated contour } C \\
&(b) \ x_{ij} \geq 0 \qquad \text{for all } (i, j) \in E \\
&(c) \ x_{ij} \text{ integer valued.}
\end{aligned} \tag{1}$$

In what follows we will show that this problem is equivalent to the linear programming problem obtained by dropping condition (c). To see this let us define $G' = (V', E')$ where

$$\begin{array}{lll}
V' = \{1, 2, \ldots, n\} \cup \{d, e\} & & \\
\text{for } 1 \leq i < j \leq n & (i, j) \in E' & \text{if } J_{ij} > 0 \\
\text{for } 1 \leq i \leq n & \begin{cases} (d, i) \in E' \\ (i, e) \in E' \end{cases} & \begin{array}{l} \text{if } F_i \geq 0 \\ \text{if } F_i < 0. \end{array}
\end{array}$$

From the max-flow min-cut theorem (cf Fulkerson 1971) we conclude that the problem of minimising

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} J_{ij} x_{ij} + \sum_{(d,i)\in E'} F_i x_{di} + \sum_{(i,e)\in E'} |F_i| x_{ie}$$

subject to

$$\begin{aligned}
&\sum_{(i,j)\in P} x_{ij} \geq 1 \qquad \text{for each path } P \text{ between } d \text{ and } e \\
&x_{ij} \geq 0 \qquad \text{for each edge } (i, j) \in E'
\end{aligned} \tag{2}$$

has an integer-valued optimal solution.

Since the graph G' has been obtained from the graph G by splitting the node $n+1$, it is clear that each frustrated contour of G corresponds to a path between d and e and vice versa; hence problem (2) is the same as problem (1) without the condition (c). The same kind of idea has been used by Barahona (1983).

In fact, problem (2) can be solved in $O(n^3)$ calculations by finding a minimum cut in G', the edges in the cut corresponding to the violated edges of G.

The linear programming problem defined by (1) without (c), has the following dual: maximise

$$\sum \{y_C \colon C \text{ is a frustrated contour}\}$$

subject to

$$\sum \{y_C: (i,j) \in C\} \leqslant J_{ij} \qquad \text{for } 1 \leqslant i < j \leqslant n$$

$$\sum \{y_C: (i, n+1) \in C\} \leqslant |F_i| \qquad \text{for } 1 \leqslant i \leqslant n$$

$$y_C \geqslant 0 \qquad \text{for each frustrated contour } C.$$

The solution of this can be obtained by finding a maximum flow in the graph G'. The amounts of flow that should be sent by each path between d and e are the values of the dual variables associated with the frustrated contours. They can be interpreted as a repartition of the violation energy among the frustrated contours. This dual information permits us to study the rigidity of the ground states as has been done in Barahona *et al* (1982).

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